Notes for AA214, Chapter 7

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Stability of Linear Systems

1. Stability will be defined in terms of ODE's and $O\Delta E$'s

   (a) ODE: Couples System

   $$\frac{d\vec{u}}{dt} = A\,\vec{u} - \vec{f}(t) \tag{1}$$

   (b) $O\Delta E$: Matrix form from applying Eq. 1

   $$\vec{u}_{n+1} = C\,\vec{u}_n - \vec{g}_n \tag{2}$$

   (c) For Example: Euler Explicit $C = [I - hA]$

Inherent Stability of ODE's

1. Stability of Eq. 1 depends entirely on the eigensystem of A.

2. λ_m-spectrum of A: function of finite-difference scheme,BC

For a *stationary* matrix A, Eq. 1 is *inherently stable* if, when $\vec{f}$ is constant, $\vec{u}$ remains bounded as $t \to \infty$. $\qquad$ (3)

3. Note that inherent stability depends only on the transient solution of the ODE's.

$$\begin{aligned} \vec{u}(t) &= c_1 \left(e^{\lambda_1 h}\right)^n \vec{x}_1 + \cdots + c_m \left(e^{\lambda_m h}\right)^n \vec{x}_m + \cdots \\ &+ c_M \left(e^{\lambda_M h}\right)^n \vec{x}_M + P.S. \end{aligned} \qquad (4)$$

4. ODE's are inherently stable if and only if

$$\boxed{\Re(\lambda_m) \leq 0 \quad \text{for all} \quad m} \tag{5}$$

5. For inherent stability, all of the λ eigenvalues must lie on, or to the left of, the imaginary axis in the complex λ plane.

6. This criterion is satisfied for the model ODE's representing both diffusion and biconvection.

Numerical Stability of $O\Delta E$'s

1. Stability of Eq. 2 related to the eigensystem of its matrix, C.

2. σ_m-spectrum of C: determined by the $O\Delta E$ and are a function of λ_m

$$
\begin{aligned}
\vec{u}_n &= c_1(\sigma_1)^n \vec{x}_1 + \cdots + c_m(\sigma_m)^n \vec{x}_m + \cdots \\
&+ c_M(\sigma_M)^n \vec{x}_M + P.S.
\end{aligned}
\tag{6}
$$

3. Spurious roots play a similar role in stability.

4. The $O\Delta E$ companion to Statement 3 is

> For a *stationary* matrix C, Eq. 2 is *numerically stable* if, when $\vec{g}$ is constant, $\vec{u}_n$ remains bounded as $n \to \infty$.
$$\tag{7}$$

5. Definition of stability: referred to as asymptotic or time stability.

6. Time-marching method is numerically stable if and only if

$$\boxed{|(\sigma_m)_k| \leq 1 \quad \text{for all } m \text{ and } k} \tag{8}$$

7. This condition states that, for numerical stability, all of the σ eigenvalues (both principal and spurious, if there are any) must lie on or inside the unit circle in the complex σ-plane.

8. This definition of stability for OΔE's is consistent with the stability definition for ODE's.

1. Our Approach leads to

   (a) The PDE's are converted to ODE's by approximating the space derivatives on a finite mesh.

   (b) Inherent stability of the ODE's is established by guaranteeing that $\Re(\lambda) \leq 0$.

   (c) Time-march methods are developed which guarantee that $|\sigma(\lambda h)| \leq 1$ and this is taken to be the condition for numerical stability.

Time-Space Stability and Convergence of OΔE's

1. A more classical view (but consistent) in the time-space sense.

   (a) The homogeneous part of Eq. 2, $\quad \vec{u}_{n+1} = C\vec{u}_n$

   (b) Applying simple recursion $\quad \vec{u}_n = C^n \vec{u}_0$

   (c) Using vector and matrix p-norms

   $$||\vec{u}_n|| = ||C^n \vec{u}_0|| \leq ||C^n|| \cdot ||\vec{u}_0|| \leq ||C||^n \cdot ||\vec{u}_0|| \qquad (9)$$

   (d) Assume that the initial data vector is bounded, the solution vector is bounded if

   $$||C|| \leq 1 \qquad (10)$$

   where $||C||$ represents any p-norm of C.

(e) This is often used as a *sufficient* condition for stability.

(f) Well known relation between spectral radii and matrix norms

    i. The spectral radius of a matrix is its L_2 norm when the matrix is normal, i.e., it commutes with its transpose.

   ii. The spectral radius is the *lower bound* of all norms.

2. The matrix norm approach and the $\sigma - \lambda$ analysis are consistent when both A and C have a complete eigensystem.

Numerical Stability Concepts: Complex σ-Plane

1. σ-Root Traces Relative to the Unit Circle

2. The $O\Delta E$ solution to the homogeneous part

$$\vec{u}_n = c_1 \sigma_1^n \vec{x}_1 + \cdots + c_m \sigma_m^n \vec{x}_m + \cdots + c_M \sigma_M^n \vec{x}_M$$

3. Semi-discrete approach leads to a relation between the σ and the λ eigenvalues.

4. Numerical stability of the $O\Delta E$ requires that σ-roots lie within unit circle in the complex σ-plane.

5. Trace the locus of the σ-roots as a function of the parameter λh

Stability in the Complex-σ Plane

1. Define $\sigma_{exact} = e^{\lambda h}$ and Separate:

   (a) Dissipation ($\lambda h = \beta < 0$) —— Convection ($\lambda h = i\omega$)

2. Plot $Real(\sigma)$ and $Imag(\sigma)$ for varying λh of both types.

a) Dissipation
b) Convection
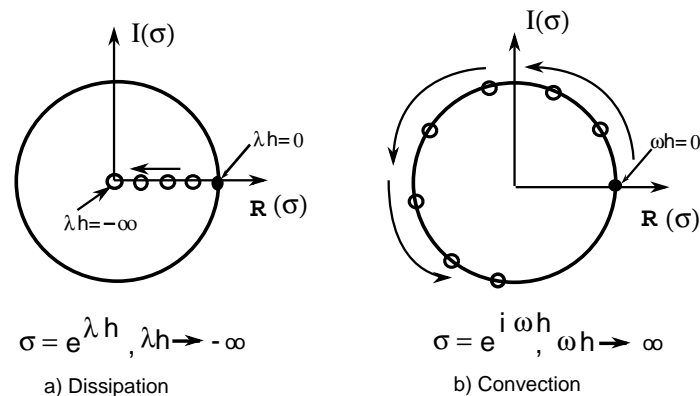
Figure 1: Exact traces of σ-roots for model equations.

$\sigma - \lambda$ Relations for Various Schemes

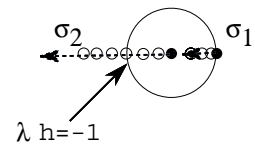|		
1.	$\sigma - 1 - \lambda h = 0$	Explicit Euler
2.	$\sigma^2 - 2\lambda h\sigma - 1 = 0$	Leapfrog
3.	$\sigma^2 - (1 + \frac{3}{2}\lambda h)\sigma + \frac{1}{2}\lambda h = 0$	AB2
4.	$\sigma^3 - (1 + \frac{23}{12}\lambda h)\sigma^2 + \frac{16}{12}\lambda h\sigma - \frac{5}{12}\lambda h = 0$	AB3
5.	$\sigma(1 - \lambda h) - 1 = 0$	Implicit Euler
6.	$\sigma(1 - \frac{1}{2}\lambda h) - (1 + \frac{1}{2}\lambda h) = 0$	Trapezoidal
7.	$\sigma^2(1 - \frac{2}{3}\lambda h) - \frac{4}{3}\sigma + \frac{1}{3} = 0$	2nd-Order Backward
8.	$\sigma^2(1 - \frac{5}{12}\lambda h) - (1 + \frac{8}{12}\lambda h)\sigma + \frac{1}{12}\lambda h = 0$	AM3
9.	$\sigma^2 - (1 + \frac{13}{12}\lambda h + \frac{15}{24}\lambda^2 h^2)\sigma + \frac{1}{12}\lambda h(1 + \frac{5}{2}\lambda h) = 0$	ABM3
10.	$\sigma^3 - (1 + 2\lambda h)\sigma^2 + \frac{3}{2}\lambda h\sigma - \frac{1}{2}\lambda h = 0$	Gazdag
11.	$\sigma - 1 - \lambda h - \frac{1}{2}\lambda^2 h^2 = 0$	RK2
12.	$\sigma - 1 - \lambda h - \frac{1}{2}\lambda^2 h^2 - \frac{1}{6}\lambda^3 h^3 - \frac{1}{24}\lambda^4 h^4 = 0$	RK4
13.	$\sigma^2(1 - \frac{1}{3}\lambda h) - \frac{4}{3}\lambda h\sigma - (1 + \frac{1}{3}\lambda h) = 0$	Milne 4th

Table 7.1. Some $\lambda - \sigma$ Relations

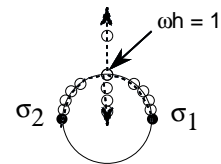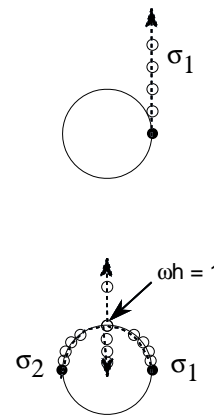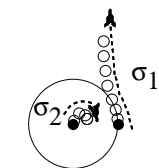# Traces of σ-roots for various methods.
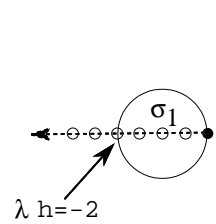
a) Euler Explicit
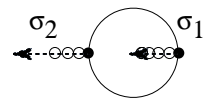
b) Leapfrog

c) AB2

Diffusion

Convection

Traces of σ-roots for various methods.

σ_1

$\lambda h = -\infty$ d) Trapezoidal

σ_3 σ_1

σ_2

e) Gazdag

σ_1

$\lambda h = -2$ f) RK2

σ_1

$\lambda h = -2.8$ g) RK4

Diffusion

σ_1

$\omega h = 2/3$

σ_2

σ_1

σ_3

σ_1

σ_1

$\omega h = 2\sqrt{2}$
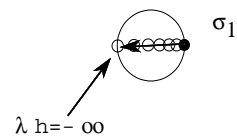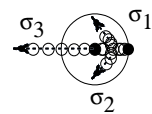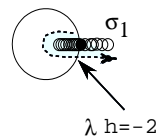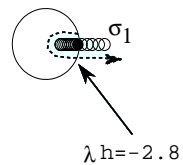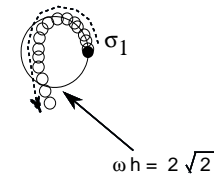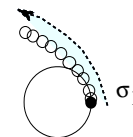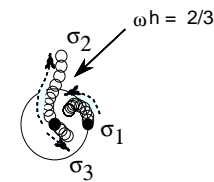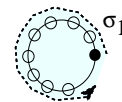
Convection

Types of Stability

1. Conditional Stability: Explicit Methods

   (a) $O\Delta E$'s where $\lambda h \leq Constant$

   (b) λ spectrum, e.g. $\lambda_b h = -\frac{ah}{\Delta x}(1 - cos(k\Delta x) + isin(k\Delta x))$

   (c) Given Δx, wave speed a, and difference scheme: λ fixed

   (d) Adjust $h = \Delta t$ to satisfy stability bound

   (e) Time accuracy: use an appropriate h

   (f) Mildly-unstable: Prof. Milton VanDyke

   Lock bike fork and peddle as fast as you can, you may cross the street before you fall over and a truck hits you.

2. Un-Conditional Stability: Implicit Methods

> A numerical method is *unconditionally stable* if it is stable for all ODE's that are inherently stable.

(a) $O\Delta E$'s where $\lambda h \to \infty$ is stable

(b) Time accuracy: use an appropriate h

(c) Steady-State: any h which converges fast.

(d) Computationally expensive compared with Explicit Methods

Stability Contours in the Complex λh Plane.

1. Another view of stability properties of a time-marching method is to plot the locus of the complex λh for which $|\sigma| = 1$

2. $|\sigma|$ refers to the maximum absolute value of any σ, principal or spurious, that is a root to the characteristic polynomial for a given λh.

3. Inherently stable ODE's lies in the left half complex-sigma plane

Example for Euler Explicit

1. Euler explicit: $\sigma_{ee} = 1 + h\lambda$

   (a) Wave equation: central differencing, $\lambda_c = -ai\frac{sin(k\Delta x)}{\Delta x}$

   $$\sigma_{ee} = 1 - \frac{ah}{\Delta x}i sin(k\Delta x)$$

   (b) $|\sigma_{ee}| > 1.0$ for all h, unconditionally unstable

2. Wave equation: 1st order backward differencing,
   $\lambda_b h = -\frac{ah}{\Delta x}(1 - cos(k\Delta x) + isin(k\Delta x))$

   (a) $|\sigma_{ee}| \leq 1.0$ for all some h, conditionally stable

   (b) Note: $CFL = \frac{ah}{\Delta x}$, CFL Number

3. Complex λ-plane, Euler explcit, $\sigma_{ee} = 1 + \lambda h$

   (a) Let $\lambda h = x + iy$, then $\sigma_{ee} = 1 + x + iy$

   $$|\sigma_{ee}| = \sqrt{(1+x)^2 + y^2}$$

   (b) Contour of $|\sigma_{ee}| = 0.8$ leads to $(1+x)^2 + y^2 = (0.8)^2$: circle in x, y centered at $x = -1$ with radius 0.8 , **Stable**

   (c) Contour of $|\sigma_{ee}| = 1.2$ leads to $(1+x)^2 + y^2 = (1.2)^{2}$': circle in x, y centered at $x = -1$ with radius 1.2, **Un-Stable**

Example for Euler Implicit

1. Euler implicit: $\sigma_{ei} = \frac{1}{1 - \lambda h}$

   (a) Wave equation: central differencing, $\lambda_c = -ai\frac{sin(k\Delta x)}{\Delta x}$

   $$\sigma_{ei} = \frac{1}{1 + \frac{ah}{\Delta x} i\, sin(k\Delta x)}$$

   (b) $|\sigma_{ei}| < 1.0$ for all h, unconditionally stable

   (c) Even for Compex λh: unconditional stability

2. Complex λ-plane, Euler Implcit, $\sigma_{ei} = \frac{1}{1-\lambda h}$

  (a) Let $\lambda h = x + iy$, then $\sigma_{ei} = \frac{1}{1-x-iy}$

$$|\sigma_{ei}| = \frac{1}{\sqrt{(1-x)^2 + y^2}}$$

  (b) Contour of $|\sigma_{ei}| = 0.8$ leads to $(1-x)^2 + y^2 = (\frac{1}{0.8})^2$: circle in x, y centered at $x = 1$ with radius $\frac{1}{0.8}$, **Stable**

  (c) Contour of $|\sigma_{ei}| = 1.2$ leads to $(1-x)^2 + y^2 = (\frac{1}{1.2})^2$: circle in x, y centered at $x = 1$ with radius $\frac{1}{1.2} < 1.0$, **Un-Stable**

  (d) The unstable contours are in the right half of the inherent stable of the ODE's
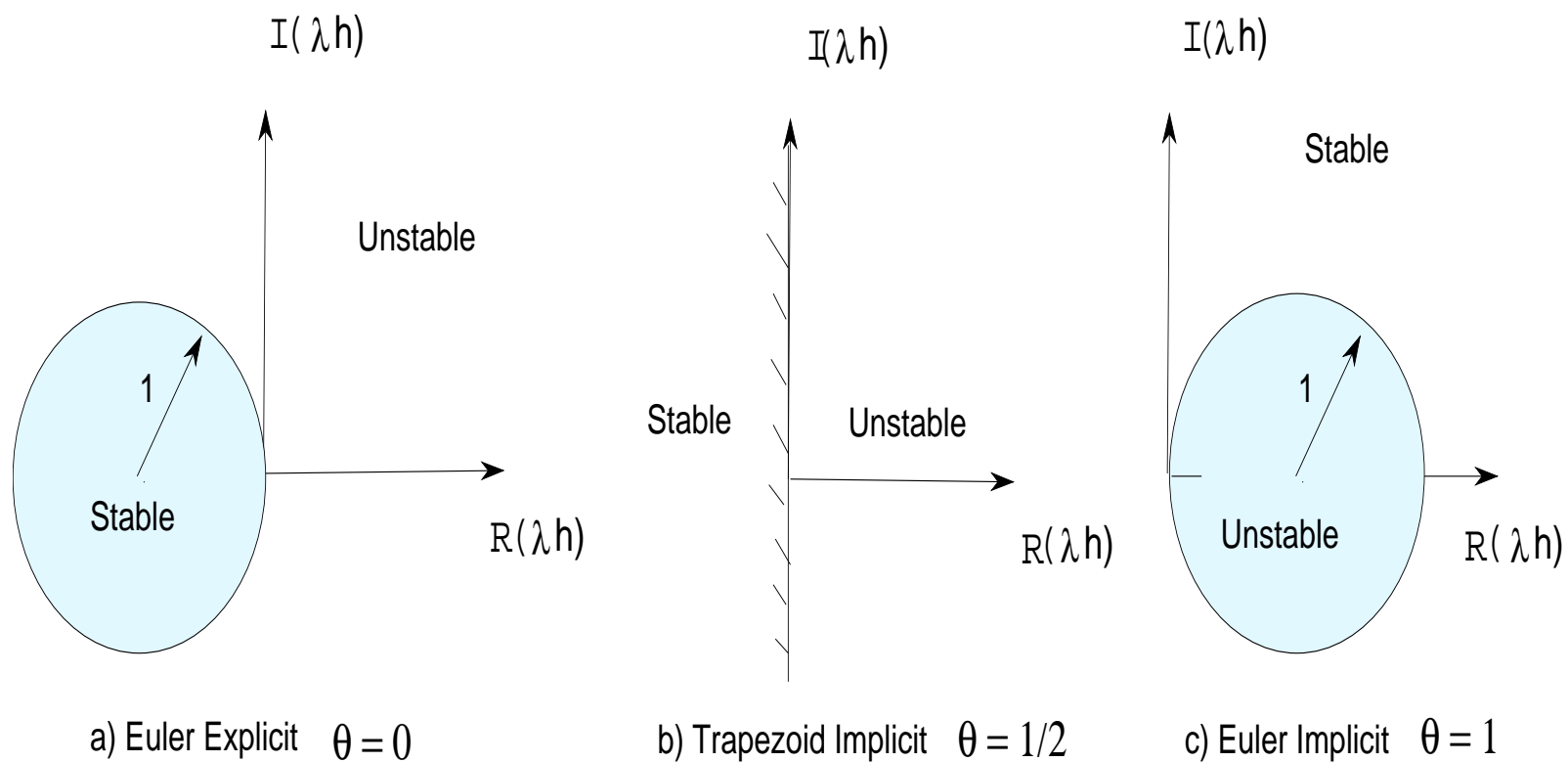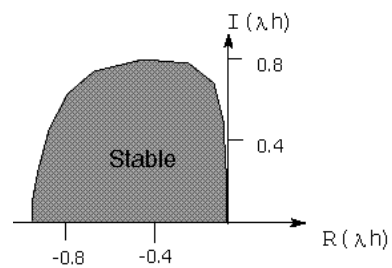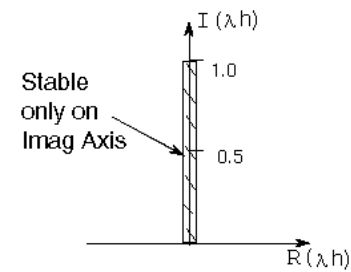
Figure 2: Stability contours for the θ-method.
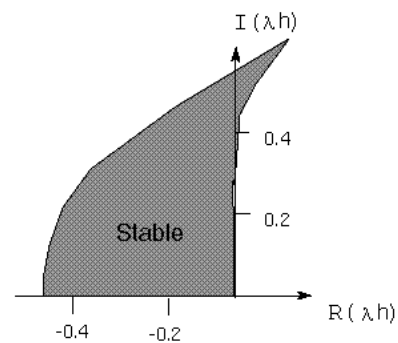
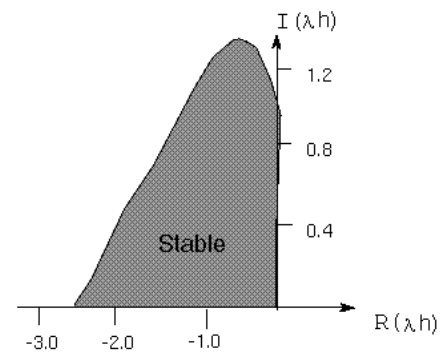# Stability contours for some explicit methods.
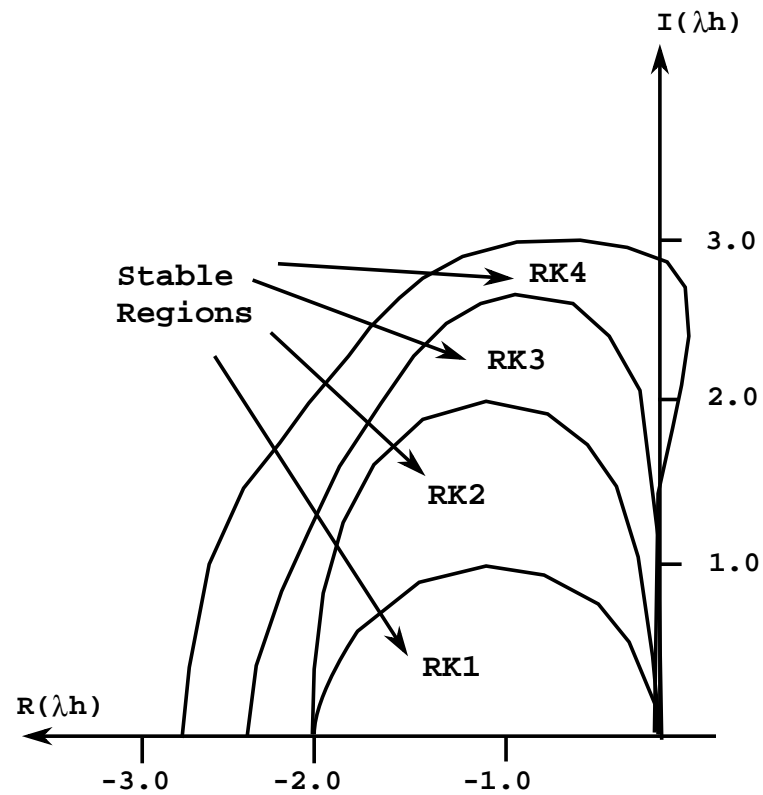

a) AB2

b) Leapfrog

c) Gazdag

d) ABM3

Stability contours for Runge-Kutta methods.

Fourier Stability Analysis

1. Classical stability analysis for numerical schemes

2. Fourier or von Neumann approach.

   (a) Periodic in space derivative, similar to modified wave number

   (b) Usually carried out on point operators

   (c) Does not depend on an intermediate stage of ODE's.

3. *Strictly speaking* it applies only to difference approximations of PDE's that produce OΔE's

4. Serves as a fairly reliable *necessary* stability condition, but it is by no means a *sufficient* one.

The Basic Procedure

1. Impose a spatial harmonic as an initial value on the mesh

2. Will its amplitude grow or decay in time?

3. Determined by finding the conditions under which

$$u(x,t) = e^{\alpha t} \cdot e^{i\kappa x} \tag{11}$$

4. Is a solution to the *difference* equation, where κ is real and $\kappa \Delta x$ lies in the range $0 \le \kappa \Delta x \le \pi$.

5. For the general term,

$$u_{j+m}^{(n+\ell)} = e^{\alpha(t+\ell\Delta t)} \cdot e^{i\kappa(x+m\Delta x)} = e^{\alpha\ell\Delta t} \cdot e^{i\kappa m\Delta x} \cdot u_j^{(n)}$$

6. $u_j^{(n)}$ is common to every term and can be factored out.

7. Find the term $e^{\alpha \Delta t}$, which we represent by σ, thus:

$$\sigma \equiv e^{\alpha \Delta t}$$

8. Since $e^{\alpha t} = \left(e^{\alpha \Delta t}\right)^n = \sigma^n$

$$\boxed{\text{For numerical stability } |\sigma| \leq 1} \qquad (12)$$

9. Solve for the σ's produced by any given method

10. A necessary condition for stability, make sure that, in the worst possible combination of parameters, condition 12 is satisfied.

Example 1

1. Finite-difference approximation to the model diffusion equation

2. Richardson's method of overlapping steps.

$$u_j^{(n+1)} = u_j^{(n-1)} + \nu \frac{2\Delta t}{\Delta x^2} \left(u_{j+1}^{(n)} - 2u_j^{(n)} + u_{j-1}^{(n)} \right) \tag{13}$$

  (a) Substitution of Eq. 11 into Eq. 13

$$\sigma = \sigma^{-1} + \nu \frac{2\Delta t}{\Delta x^2} \left(e^{i\kappa\Delta x} - 2 + e^{-i\kappa\Delta x} \right)$$

or

$$\sigma^2 + \underbrace{\left[\frac{4\nu\Delta t}{\Delta x^2} (1 - \cos\kappa\Delta x) \right]}_{2b} \sigma - 1 = 0 \tag{14}$$

  (b) Eq. 11 is a solution of Eq. 13 if σ is a root of Eq. 14.

(c) The two roots of Eq. 14 are

$$\sigma_{1,2} = -b \pm \sqrt{b^2 + 1}$$

(d) One $|\sigma|$ is always > 1.

(e) Therefore, that by the Fourier stability test, Richardson's method of overlapping steps is unstable for all ν, κ and Δt.

1. Finite-difference approximation for the model biconvection equation

$$u_j^{(n+1)} = u_j^{(n)} - \frac{a\Delta t}{2\Delta x}\left(u_{j+1}^{(n)} - u_{j-1}^{(n)}\right) \tag{15}$$

$$\sigma = 1 - \frac{a\Delta t}{\Delta x} \cdot i \cdot \sin \kappa \Delta x$$

2. $|\sigma| > 1$ for all nonzero a and κ.

3. Thus we have another finite-difference approximation that, by the Fourier stability test, is unstable for any choice of the free parameters.